Biomedical Signal Processing and Control 11 (2014) 42–52

Contents lists available at ScienceDirect

Biomedical Signal Processing and Control

journal homepage: www.elsevier.com/locate/bspc

A survey of methods used for source localization using EEG signals

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A R T I C L E I N F O

Article history:
Received 19 July 2013
Received in revised form
20 December 2013
Accepted 20 January 2014
Available online 28 March 2014

A B S T R A C T

The EEG source localization which is used to localize the electrical activity of brain has been an active area of research as it provides useful information for study of brain's physiological, mental and functional abnormalities. This problem is called EEG inverse problem. The localization of the active sources needs the solution of ill posed EEG inverse problem. Since the foundation of this field till today, many methods have been developed with the aim of in-depth localization, high resolution, reduction in localization/energy error and decreased computational time. In this survey, EEG inverse problem is discussed with its primary to most developed and recent solutions. The introduction to the field along with the categorization of different solutions is provided. Also, the relative advantages and limitations for each method are discussed. Finally, the challenges and future recommendations are provided, in the end, for further improvement of EEG inverse problem in terms of resolution, computational power and localization error.

Contents

1. Introduction

Functional brain imaging is a multidisciplinary research field that deals with the non-invasive brain imaging techniques. These techniques are used for better understanding of electrophysiological, hemodynamic, metabolic and neurochemical process that describe normal and pathological brain functionalities [1]. These

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http://dx.doi.org/10.1016/j.bspc.2014.01.009

techniques include single photon emission computer tomography (SPECT), positron emission tomography (PET), functional magnetic resonance imaging (fMRI), magneto encephalography (MEG) and electroencephalography (EEG) [2]. These imaging methods have variant temporal and spatial resolution depending upon their specific application. The functionalities of these imaging techniques are used for clinical applications for improved understanding and treatment of neurological and neurophysiological disorders such as epilepsy, schizophrenia, depression and Alzheimer's diseases. Among these diseases, epilepsy is the most important and common neurological disorder as 1% of the world population is suffering from it [3]. Since the epileptic activity propagates very fast, several hyper regions are seen in the images; therefore a method with high temporal resolution is required to cope with this problem. To overcome this problem, EEG is regarded as best non-invasive diagnosis tool used in epilepsy surgery centres due to its high temporal resolution in order of milliseconds [4].

EEG is the neuroimaging technique which was developed by German Physicist Hans Berger in 1929. EEG produces the measurements of a set of potential differences between pair of electrodes, when these electrodes are placed on the scalp [5–7]. Berger was interested in cerebral localization particularly for localizing the tumours in brain [8]. However, Kornmuller was the first Neuro Scientist who discovered the significance of using multichannel recordings for the coverage of a wider brain region.

EEG can be defined as: "the non-invasive/invasive neuroimaging technique having high temporal, low spatial resolution which records the brain activity by measuring electrical signals generated with the help of electrodes placed on the scalp in order to diagnose/analyse different neural disorders (epilepsy, tumours, locating head damages, etc.)". The EEG recordings can be used for direct, real time, monitoring of spontaneous and evoked brain activity which allow for spatiotemporal localization of neuronal activity [9].

The estimation of location and distribution of the current sources responsible for the electromagnetic activity inside the brain based on the potential recorded through the electrodes is one of the major problems in EEG. This problem is termed as brain source localization or EEG inverse problem as the data (potentials) is given and one has to design the model from the available data. In other words, given a set of electric potentials from discrete sites on the surface of the head and the associated positions of those measurements and the geometry and conductivity of regions within head, the location and magnitude of the current sources within the brain is calculated [10].

This source modelling by EEG for non-invasive localization of epileptogenic zones helps in clinical applications such as for surgery in patients with partial seizures [11]. The localization information for the active sources in the brain helps to diagnose pathological, physiological, mental and functional abnormalities related to the brain. Therefore, EEG source localization has been an active area of research since decades. In the past few years, source localization method due to its application in the clinical applications for the epileptic surgery has produced more than 150 research publications [12]. These publications are based upon software based mathematical solution for EEG inverse problem. However, less than half of these publications addressed the issue of clinical validation for investigation of focal epilepsy [13].

This survey is carried out to study the basic concepts behind EEG inverse problem and its solution through various proposed algorithms. This survey begins with basic understanding of mathematical background related to inverse problem in general. The existing algorithms for solution of EEG inverse problem such as minimum norm, LORETA, sLORETA, eLORETA, MUSIC, FOCUSS and ICA are discussed thoroughly to provide a clear view of these methods to the reader. A detailed discussion followed by mathematical interpretation and physical meaning for each algorithm as well as its advantages and limitations are presented. The listing provided helps the reader to get an overview and deeper understanding of each method. Apart from this, the review also provides a generalized comparison between various algorithms in terms of resolution, computational complexity, and validation and localization error.

2. Background concepts

Localization of active sources of brain is termed as EEG source localization. This process involves the prediction of scalp potentials from the current sources in the brain (forward problem) and the estimation of the location of the sources from scalp potential measurements (termed as inverse problem) [14]. The efforts to understand the localization problem began 40-years ago by correlating the existing body of electro physiological knowledge about the brain to the basic physical principles controlling the volume currents in conductive media [15–20].

2.1. Forward and inverse problem

In the physical world, if the data values are extracted/estimated from the given model with the help of some physical theories being applied to the model, then the problem is said to be modelization problem, simulation problem or forward problem [21]. This is a straight-forward procedure which requires fewer computations with fewer errors because the model with complete description is with us. However, the inverse problem suggests predicting the model with the help of the available measured parameters.

In the physical world, a finite amount of data is available to reconstruct a model with infinitely many degrees of freedom. Hence, the inverse problem is not unique and there are many models that can explain the data equally well. On the contrary, Forward problem has a unique solution. As an example taken from [21], consider measurements of the gravity field around a planet: given the distribution of mass inside the planet, we can uniquely predict the values of the gravity field around the planet (forward problem). However, there are different distributions of mass which can give exactly the same gravity field in the space outside the planet. Therefore, the inverse problem of inferring the mass distribution from observations of the gravity field has multiple solutions (in fact, an infinite number). Because of this, in the inverse problem, one needs to make explicit any a priori information on the model parameters. One also needs to be careful in the representation of uncertainties in the data.

The inverse problem has got non-uniqueness in nature which means many models can fit the data. Also, the estimated data is tainted with errors, therefore the estimated model always differ from the true model. The model with finite degrees of freedom is termed as discrete model and one with continuous data and infinite degrees of freedom is termed as continuous model. The model estimation and model appraisal are different for both systems. According to Hadamard [22], if a physical problem has got a solution with uniqueness and stability in it then the inverse problem is assumed to be well posed; otherwise it is termed as ill-posed. Majority of geophysical problems are ill posed which means they have got non-uniqueness and instability. Therefore, it assumes that the predicted model is just an approximation of the true model. The inversion problem consists of two steps, i.e., estimation problem and appraisal problem [23]. Let the true data be denoted by $\boldsymbol{d}$, true model denoted by $\boldsymbol{m}$ and the estimated model by $\tilde{\boldsymbol{m}}$ and data being tainted by error. Then one can assume the Inverse problem as combination of what is to estimate and a relationship between estimated and true which is said to be appraisal.

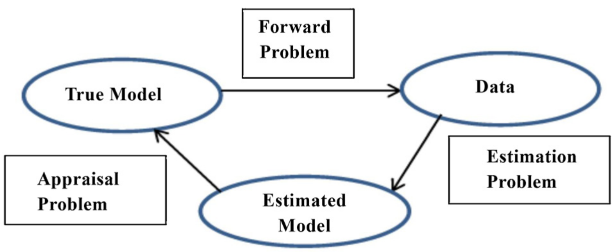

Fig. 1. Pictorial diagram of inversion for a physical model [23].

So,

Inverse Problem = Estimated Problem + Appraisal Problem

2.2. Model estimation

Let a physical model with finite dimension coupled with model parameters be ordered in a vector $\boldsymbol{m}$ and the related data being ordered in $\boldsymbol{d}$ vector, then the so called theory operator $\boldsymbol{A}$ is used to relate the data vector $\boldsymbol{d}$ and model vector $\boldsymbol{m}$ as [23]:

$$\boldsymbol{d} = \boldsymbol{A}\boldsymbol{m} + \boldsymbol{e} \tag{1}$$

where $\boldsymbol{e}$ is the error tainted in the original data. As discussed earlier that the true model $\boldsymbol{m}$ is differing the estimate model $\mathbf{m}'$ which leads to the conclusion that the linear mapping from the data to be estimated model as [23]:

$$\tilde{\boldsymbol{m}} = \boldsymbol{A}^{-g}\boldsymbol{d} \tag{2}$$

where is $\boldsymbol{A}^{-g}$ is the generalized inverse of the matrix $\mathbf{A}$. However, if one wants to draw a relationship between the true model $\boldsymbol{m}$ and the estimated model $\tilde{\boldsymbol{m}}$, then,

$$\tilde{\boldsymbol{m}} = \boldsymbol{A}^{-g}\boldsymbol{A}\boldsymbol{m} + \boldsymbol{A}^{-g}\boldsymbol{e} \tag{3}$$

where the matrix $\boldsymbol{R} = \boldsymbol{A}^{-g}\boldsymbol{A}$ is known as resolution kernel. The resolution kernel is used to quantify the amount of retrieval of model parameters estimated. For perfect resolution, the resolution kernel is equal to identity matrix [23].

Rewriting Eq. (3) we have:

$$\tilde{\boldsymbol{m}} = \boldsymbol{m} + (\boldsymbol{A}^{-g}\boldsymbol{A} - \boldsymbol{I})\boldsymbol{m} + \boldsymbol{A}^{-g}\boldsymbol{e} \tag{4}$$

Ideally, the resolution kernel $\boldsymbol{R}$ should be equal to identity matrix so as to approximate the estimated model to the true model as near as possible with error propagation (last term of Eq. (3)) zero value. Another thing worth to be noted here is that the error propagation quantity is not deterministically known rather one has to use statistical analysis to analyse the errors present in the data (Fig. 1).

The EEG source localization is an underdetermined ill-posed inverse problem due to the fact that number of unknown parameters is greater than number of known parameters. There exist two general approaches for the localization as proposed by researchers. Either the signals are assumed to be generated by a small number of focal sources. This approach is called as equivalent current dipole (ECD). However, if all possible source locations are assumed simultaneously, then it is known as linear distributed approach [24]. There are many existing methods for the solutions of EEG inverse problem which can be categorized according to the methodology adopted for implementation of each. Some core methods are defined independently; whereas, other methods are hybrid in nature. Fig. 2 shows the categorization for the various inverse methods for source localization using EEG signals.

For the EEG inverse problem; there exist N_E instantaneous measurements and N_V voxels in the brain. The voxels can be determined by uniformly dividing the solution space. Each voxel has got a point source which may be a vector with three unknown components (i.e., the three dipole moments), or a scalar (unknown dipole amplitude, known orientation). Hence mathematically, the equation relating scalp potentials and current density in vector/matrix form is given as [24]:

$$\phi = KJ + c_1 \tag{5}$$

where $\phi \in R^{N_E \times 1}$ is vector containing measurements of potential differences taken from N_E electrodes with a reference electrode, $K \in R^{N_E \times (3N_V)}$ is the lead field matrix corresponding to N_V voxels, $J \in R^{(3N_V) \times 1}$ is the current density, $1 \in R^{N_E \times 1}$ and C is an arbitrary constant.

The structure of lead field matrix K can be defined as:

$$K = \begin{bmatrix} k_{11}^T & k_{12}^T & \cdots & k_{1N_v}^T \\ k_{21}^T & k_{22}^T & \cdots & k_{2N_v}^T \\ \vdots & \vdots & \ddots & \vdots \\ k_{N_E1}^T & k_{N_E2}^T & \cdots & k_{N_EN_v}^T \end{bmatrix}$$

where $k_{ev} \in R^{3 \times 1}$ (for $e = 1, \ldots, N_e$ and for $v = 1, \ldots, N_v$) corresponds to the scalp potentials at the eth electrode due to three orthogonal unit strength dipoles at voxel v.

Also, the current density is expressed as:

$$J = \begin{bmatrix} j_1 \\ j_2 \\ \cdots \\ j_{N_V} \end{bmatrix}$$

where $j_v \in R^{3 \times 1}$ can be defined as current density at vth voxel.

2.3. Mathematical representation for inverse solution

The mathematical formulation for the linear solution of an instantaneous, 3D, discrete EEG inverse problem can be written as [24]:

$$J = T\phi \tag{6}$$

where T is generalized inverse of K such that, $KT = H_N$. Here H_N is average reference operator and is denoted and defined as [23]:

$$H_N = I_N - \frac{1}{N} 1_N 1_N^T \tag{7}$$

where I_N denotes the $N \times N$ identity matrix and 1_N is a $N \times 1$ matrix composed of ones.

The resolution kernel, $\boldsymbol{R}$, as defined earlier for the generalized inverse problem, also is defined for the solution of discrete, 3D, instantaneous EEG inverse problem. It checks the quality of certain algorithm that how much the given algorithm correctly estimates the current density J. Following is the relationship between the true (J) and estimated current density (J'):

$$J' = \boldsymbol{R}J \tag{8}$$

From above discussion it can be summarized that:

(1) The localization algorithm should be able to build a model/tomography with a minimum of localization error.

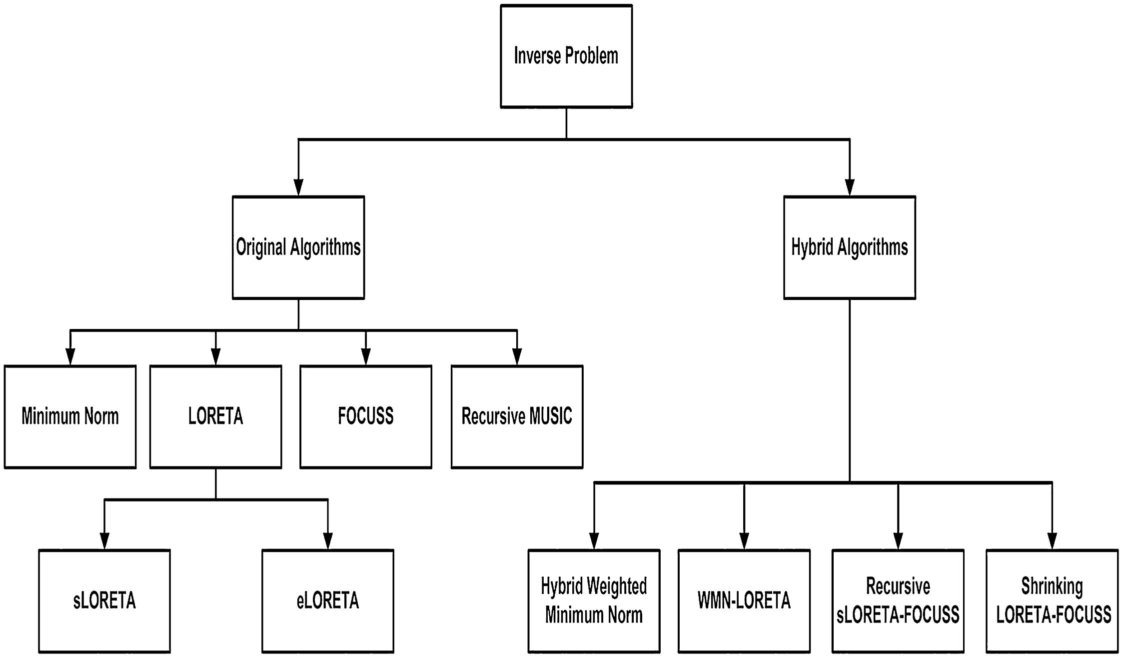

Fig. 2. Flow diagram of inverse methods used for EEG source localization.

(2) Due to non-uniqueness of inverse problem, perfect tomography is difficult; however, hope goes for less erroneous model as developed by using MUSIC, RAP-MUSIC LORETA, eLORETA, sLORETA, FOCUSS, etc. algorithms.
(3) The initial step for inverse solution requires the calculation of forward model with the assumption of one or more dipoles.
(4) The forward solution from the dipole model is predicted for the approximation of potential.
(5) The location and orientation of dipole are estimated by minimizing the least square error between the calculated and measured potential.
(6) The realistic head modelling can improve the accuracy for forward solution.
(7) Mostly algorithms use four shell concentric model which includes scalp, skull, cerebrospinal fluid (CSF) and brain for inverse modelling.
(8) For the forward problem, potential ϕ is unknown while the lead field matrix K and current density is known. However, for inverse solution, the current density J is unknown.

For an ideal, noise free result, one can predict that the resolution matrix should be equal to identity matrix so that the estimated and true current densities are same. There are so many solutions provided to instantaneous, distributed, discrete, linear EEG inverse problem starting from Hamalainen and Ilmoniemi in 1984 by writing an article titled as “Interpreting measured magnetic fields of the brain: estimates of current distributions” [25]. After this, there started an era for development of different localization methods with various techniques and algorithms. These methods include minimum norm, weighted minimum norm (WMN), low resolution brain electromagnetic tomography (LORETA), standardized LORETA, recursive multiple signal classifier (MUSIC), recursively applied and projected MUSIC (RAP MUSIC), shrinking LORETA–FOCUSS, hybrid weighted minimum norm method, recursive sLORETA–FOCUSS, standardized shrinking LORETA–FOCUSS (SSLOFO), etc.

3. 3D EEG source localization methods

3.1. Minimum norm

This solution for the distributed, discrete and instantaneous EEG inverse problem was suggested by Hamalainen and Ilmoniemi in 1984 [25]. The EEG inverse problem is solved by proposing the linear combination of magnetometer lead fields as an estimate for current distribution. The lead field L_i of a magnetometer at location i can be defined as:

$$B_i(J) = \int L_i(r).J(r)dV \tag{9}$$

One can express the linear relationship between the magnetometer readings, current distribution and the lead fields as following:

$$B = LJ \tag{10}$$

Hence the shortest current vector required to explain the magnetometer output can be defined by multiplying the output vector B by the pseudo inverse of L such that:

$$\hat{J} = L^{+}B \tag{11}$$

where $L^{+} = L^{T}(LL^{T})^{+}$ denotes the Moore–Penrose generalized inverse [26]. The minimum norm solution was predicted for the pure signals, signals contaminated by noise and the smoothed noisy signals. The proposed method estimated the sources with 1-cm resolution near the central sulcus, about 4 cm from the midline. However, it was suggested for the improved performance of minimum norm solution to provide some a priori information or assumption such as confining the integration area within the cortex. Also, with increased number of magnetometers, better localization can be achieved.

Though the minimum norm provides good results in terms of resolution and current estimation but it fails to address the issue of deep source localization in the outermost cortex. This occurs because minimum norm solution for EEG/MEG is a harmonic function, i.e., $\nabla^2 J = 0$; as the harmonic functions attain maximum values at the boundaries of their domain; which in this case is outermost cortex. Also, upon comparison with newer techniques such as LORETA and WMN, minimum norm solution has got more localization error with disadvantage of incapability of localizing non-boundary sources [27].

3.2. Low resolution electromagnetic tomography (LORETA)

Introduced and defined by Pascual-Marqui in 1994 in [28], LORETA computes the current distribution throughout full brain volume. This method assumes the simultaneous and synchronous excitation of neighbouring neurons, i.e., the current density at any given point on the cortex is maximally similar to the average current density of its neighbour. The discrete, 3D distributed, linear inverse solution is provided with much better time resolution but low spatial resolution.

The generalized inverse problem for the LORETA can be defined and explained mathematically as:

$$\min_J F_W \tag{12}$$

where

$$F_W = \left\| \Phi - KJ \right\|^2 + \alpha J^T W J \tag{13}$$

In the above equation, the Tikhonov parameter $\alpha > 0$ is the control parameter used for controlling of relative importance between penalty for being unfaithful to the measurements and a penalty for a large current density norm [23].

The solution is

$$\hat{J}_W = T_W \phi \tag{14}$$

The value of T_W can be calculated by:

$$T_W = W^{-1} K^T \left(K W^{-1} K^T + \alpha H \right)^+ \tag{15}$$

The weight matrix $W \in R^{(3N_v)\times(3N_V)}$ is used to implement the discrete spatial Laplacian operator with the help of B. For LORETA, the weight matrix W is calculated as:

$$W = (\Omega \otimes I_3) B^T B (\Omega \otimes I_3) \tag{16}$$

where

$$\Omega_{\beta\beta} = \sqrt{\sum_{\alpha=1}^{N} k_{\alpha\beta}^T k_{\alpha\beta}} \tag{17}$$

for $\beta = 1, \ldots, M$ and $\otimes$ defines Kronecker product and B is the matrix which implements the discrete spatial Laplacian operator for smoothness in the inverse solution. B can be defined as:

$$B = \frac{6}{d^2}(A - I_{3M}) \tag{18}$$

$$A = A_0 \otimes I_3, A_0 = \frac{1}{2}(I_M + [diag(A_1 1_M)]^{-1}) A_1$$

$$[A_1]_{\alpha\beta} = \left\{ \begin{matrix} \frac{1}{6}, & \text{if} \left\| v_\alpha - v_\beta \right\| = d \\ 0 & \end{matrix} \right\}, \quad \forall \alpha, \beta = 1, \ldots, M \tag{19}$$

In the set of equations defined above, the inverse matrix B^{-1} implements the discrete spatial smoothing operator and d is the minimum inter-grid-point distance, the $diag(A_1 1_M)$ is diagonal matrix whose entries are explained from the matrix $A_1 1_M$. The set of the equation provided explains the Laplacian operator used to implement LORETA. LORETA provides smooth and better localization for deep sources with less localization errors but with low spatial resolution and blurred localized images of a point source with dispersion in the image. The low spatial resolution of LORETA is undesirable in some cases such as feature extraction of spatio-temporal pattern recognition where high resolution is needed. Also, it is shown in [28] that LORETA has high localization capability for localizing the boundary sources as discussed that out of 819 cases, LORETA localizes 383 cases with zero localization error (47%). There are some modifications done on this basic localizing technique which include sLORETA, eLORETA and some hybrid algorithms such as shrinking LORETA–FOCUSS, standardized shrinking LORETA–FOCUSS, WMN–LORETA, recursive sLORETA–FOCUSS, etc. which shall be discussed subsequently in this paper.

3.3. Focal under determined system solution (FOCUSS)

This tomographic reconstruction technique for the solution of Ill posed EEG/MEG Inverse problem was proposed and explained in [29]. FOCUSS is high resolution non-parametric technique which uses the forward model that assigns the current to each element within a predetermined reconstruction region. It is recursive in nature that is the weights are iterated at each step from the solution of previous step. The mathematical calculations for the recursive steps in FOCUSS are done with the help of weighted minimum norm method. The expression for computation of unknown current element I can be given as:

$$I = W(GW)^+ B = WW^T G^T \left(GWW^T G^T \right)^{-1} B \tag{20}$$

where W is a dimensionless $n \times n$ matrix which can be altered to produce recursive schemes. It can be reconstructed by taking its diagonal elements to be the previous iterative step solution as:

$$W_k = \begin{bmatrix} I_{1K-1} & \cdots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \cdots & I_{nk-1} \end{bmatrix}$$

where I_{ik-1} represents the ith element of the vector I at the $(k-1)$th iteration, and k is the index of the iteration step. The next weight matrix can be calculated by just multiplying W_{k-1} by W_k to get new matrix. It can be observed that the diagonal entries of weight matrix correspond to current elements. With the help of mathematical and theoretical concepts just described above, simulations and comparisons are carried out between true, minimum norm, unbiased minimum norm and FOCUSS algorithm for near-surface, mid-depth sources and deep sources in [29].

The FOCUSS algorithm provides better localization capability as compared with different algorithms and is able to handle non-uniquely defined localized energy sources. Also, FOCUSS algorithm has got better spatial resolution and is stable whenever subjected to any change.

3.4. Recursive multiple signal classification (MUSIC)

This algorithm was proposed in [30], in which a single dipole is scanned through a grid confined to a 3D head or source volume. The forward model for the dipole at each grid point is projected against a signal subspace which is calculated from the EEG measurements. The sources are located where the projection is best onto the signal subspace. However, one of the major problems with the MUSIC approach is the selection of location that can provide the best projection in the practical case as there exists noise and error in signal subspace and forward model. Recursive MUSIC algorithm which is a modification in MUSIC algorithm can combat with the limitations of MUSIC in terms of localizing synchronous sources through the use of spatio-temporal independent topographies (IT) model.

The mathematical relationship relating IT model and signal subspace can be defined as [30]:

$$F = AS^T + E \tag{21}$$

where E is random error matrix comprising n time slices such that $E \equiv [e(t_1), \ldots, e(t_n)]$ added to the data $\overset{\approx}{F} = A(\rho, \theta)S^T$, to produce F. The purpose is to determine dipole location (ρ) and dipole orientation (θ) when F is given. Hence the parameter estimation requires some mathematical manipulation of the true and estimated data by taking noise into consideration. Hence,

$$\{\hat{\rho}, \theta\}_{ls} = arg\max\{||U_A^T \hat{\phi}_s \hat{\Sigma}_s||_F^2 + ||U_A^T \hat{\phi}_e \hat{\Sigma}_e||_F^2\} \tag{22}$$

where ρ is the set of dipole locations, θ is set of dipole orientations, U_A is the matrix whose columns are the left singular vectors of A that correspond to its non-zero singular values, contains eigenvectors such that span(ϕ_s) = span(A), (it refers to signal subspace), is the weighted sum of the projections of the estimated signal subspace eigenvectors and Σ_e is the weighted sum of the errors. The recursive MUSIC algorithm and its modifications such as RAP MUSIC [31] provide good localization with less complexity. Also the algorithm is extendable with straightaway procedure rather than a sequential way.

3.5. Hybrid weighted minimum norm

This algorithm was produced in [32] as a hybrid algorithm with initial reconstruction carried out with the help of LORETA algorithm and then iterative calculations by using FOCUSS algorithm. The LORETA algorithm is applied as it can localize the real current density distribution under the frame of weighted minimum norm which provides smooth solution.

The discrete result for the inverse problem in this algorithm with the initialization of LORETA can be suggested as:

$$\min_J ||BWJ|| \text{under constraint}: \quad V = KJ \tag{23}$$

where V is $N \times 1$ matrix comprising of potential differences, J contains current densities at M points within brain volume, K is lead field matrix defining the relationship between the scalps measurements and the current densities. B is discrete Laplacian operator. The weighted matrix W can be defined as:

$$W = \Omega \otimes I \tag{24}$$

Here I is identity matrix and $\otimes$ denotes the Kronecker product with Ω as $M \times M$ diagonal matrix with diagonal elements as:

$$\Omega_{ii} = \sqrt{\sum_{\alpha=1}^{N} k_{\alpha i}^T k_{\alpha i}} \tag{25}$$

Hence the unique solution for the method can be described mathematically as:

$$\hat{J} = (WB^TBW)^{-1}K^T(K(WB^TBW)^{-1}K^T)^+V \tag{26}$$

where $\hat{J}$ is the estimated current density with all parameters having same definitions as defined earlier. A^+ is the Moore–Penrose inverse operator. If the regularization term (λ) is included in the above approximation expression for the current density for more stability and less jamming, then one can rewrite above expression as:

$$\hat{J} = (WB^TBW)^{-1}G^T(G(WB^TBW)^{-1}G^T + \lambda H)^+V \tag{27}$$

The weighted iterative method is adopted for forward problem to assure the strengthening of the grid's energy in the solution space. Hence, the weighted matrix W_K for the kth step can be calculated by solution J_{k-1} of the $(k-1)$th step as:

$$W_K = diag(J_{k-1}) \tag{28}$$

By the application of weighted minimum norm algorithm, the solution for kth iteration is given as:

$$J_K = W_K(KW_K)^+V \tag{29}$$

The results in [32] are provided by using four shell (brain, CSF, skull, scalp) spherical head model with corresponding electrical conductivities. The solution space has got the radius of 0.84 with 729 grid points within it. The method provides better results with initialization of LORETA and then iteration of weight matrix. The LORETA provides rough estimation of sources, and then the iterations make the results accurate, in-depth localized with minimized errors and good estimation.

The disadvantage with this hybrid algorithm is repeated iterations which increases computational complexity of the algorithm. Also due to continuous iterations, there exists a chance for loss of information and induction of noise.

3.6. sLORETA

sLORETA is based upon the assumption of the standardization of the current density which implies that not only the variance of the noise in the EEG measurements is taken into account but also the biological variance in the actual signal is considered [33]. This biological variance is assumed as independent uniformly distributed across the brain resulting in a linear imaging localization technique having exact, zero-localization error. This localization technique has got resemblance to the method provided by Dale et al. [33] in which the localization is provided on a standardization of the estimates of current density. However, unlike the [34], sLORETA takes into account both variations due to actual sources and noisy measurements if they exist.

The current density estimates are given by minimum norm method as in [34] with the localization inference based on standardized values of the current density estimates. The standardization for sLORETA is performed in a quite different way as compared to Dale's method resulting in zero-localization for the sLORETA.

The mathematical formulation for sLORETA is given as under:

$$F = \left\|\phi - KJ - c1\right\|^2 + \alpha \left\|J\right\| \tag{30}$$

where ϕ = electrical potentials, K is lead field matrix, J is current density, $\alpha \geq 0$ is regularization parameter. This functional has to be minimized with respect to J and c, for given K, ϕ and α. By using average reference transforms of ϕ and K, the above equation can be rewritten as:

$$F = \left\|\phi - KJ\right\|^2 + \alpha \left\|J\right\| \tag{31}$$

With minimum $\hat{J} = T\phi$ where,

$$T = K^T\left[KK^T + \alpha H\right]^+ \tag{32}$$

Therefore, for the standardized estimates of current density, the variance of estimated value of $\hat{J}$ is to be calculated. So the electric potential variance $S_\phi \in \Re^{N_E \times N_E}$ can be explained as:

$$S_\phi = KS_JK^T + S_\phi^{Noise} = KK^T + \alpha H \tag{33}$$

From the above equation, the variance for the estimated current density can be given:

$$S_{\hat{J}} = TS_\phi T^T = T(KK^T + \alpha H)T^T = K^T[KK^T + \alpha H]^+K \tag{34}$$

The sLORETA linear imaging method is:

$$\sigma_v = [S_{\hat{j}}]_v^{-(1/2)} \hat{j}_v \tag{35}$$

where $[S_{\hat{j}}]_v \in \Re^{3\times3}$ is the vth 3×3 diagonal matrix in $S^{\hat{j}}$ and $[S_{\hat{j}}]_v^{-(1/2)}$ is the symmetric square root inverse. The squared norm of σ_v corresponds to the estimate of standardized current density power as:

$$\sigma_v^T \sigma_v = \hat{j}_v^T [S_{\hat{j}}]_v^{-1} \hat{j}_v \tag{36}$$

The simulations are carried out by using Talairach human brain atlas. A total of 6430 voxels at 5 mm spatial resolution were produced under these constraints. For each dipole, there exist three unknown values making the number of unknowns as $3 \times 6430 = 19{,}290$ with 25 electrodes. Different localization methods are compared with sLORETA which include minimum norm and proposed by [35] in terms of localization errors and spatial spread. The simulations with noise and without noise demonstrate that sLORETA has far better quality with exact localization and zero-error localization as compared with minimum norm and Dale methods which shows that the sLORETA is perfect first order localization technique.

3.7. eLORETA

There have been many useful attempts to minimize the localization error by choosing the weight matrix in a more adequate way. However, there exists one methodology to give more importance to the deeper sources with reduced localization error which is termed as eLORETA. The study carried out in [35] shows performance of eLORETA by achieving depth weighting with reduced localization error from 12 to 7 mm. This method was developed and recorded as working project in the University of Zurich in March 2005 [36]. According to [36], eLORETA is a genuine inverse solution which provides exact localization with zero error in the presence of measurement and structured biological noise. Hence the family of linear imaging methods are parameterized by a symmetric matrix $C \in \Re^{N_E \times N_E}$, such that,

$$\hat{j}_i = [(K_i^T C K_i)^{-1/2} K_i^T C]\phi \tag{37}$$

where $\hat{j}_i \in \Re^{3\times1}$ is an estimator for calculation of neuronal activity at the ith voxel. In this research [35], the localization ability of a linear imaging method is elaborated by considering the actual source as an arbitrary point test source at jth voxel which assumes that:

$$\phi = K_j A \tag{38}$$

where K_j is lead field matrix and $A \in \Re^{3\times1}$ is a vector which contains dipole moments for the sources. By making use of above equations, one can write for the estimation values as:

$$\left\|\hat{j}_i\right\|^2 = A^T K_j^T C K_i (K_i^T C K_i)^+ K_i^T C K_j A \tag{39}$$

Now, considering the case of eLORETA, the current density estimator at the ith voxel can be written as:

$$\hat{j}_i = W_i^{-1} K_i^T (K W^{-1} K^T + \alpha H)^+ \phi \tag{40}$$

Upon comparison of the equations given above, one can deduce that the exact, zero error localization can be achieved with weights satisfying the equation given below:

$$W_i = [K_i^T (K W^{-1} K^T + \alpha H)^+ K_i]^{1/2} \tag{41}$$

The eLORETA method is standardized which implies that it is theoretical expected variance is unity. The simulations for the validation of this method were carried out under free academic eLORETA-KEY software with data available at [36]. The results show that eLORETA is authentic localizing method with no localization bias, which provides zero error localization in the case of non-ideal conditions, i.e., the presence of structured biological noise.

3.8. WMN–LORETA

This hybrid algorithm explained in [37] makes use of WMN and LORETA. For WMN–LORETA, the weighted minimum norm (WMN) method is used to initialize LORETA algorithm. For the weighted minimum norm method, the forward problem can be written as:

$$V = KJ = \sum_{i=1}^{3M} K_i J_i = \sum_{i=1}^{3M} \frac{K_i}{W_i} . (W_i J_i) \tag{42}$$

where

$$W_i = \left(\frac{1}{N_e}\right) \cdot \sqrt{\sum_{j=1}^{Ne} K_{ij}^2} \tag{43}$$

Therefore, for the inverse problem, the current density can be estimated as:

$$J_{\text{WMN}} = W^{-2} K^t (K W^{-2} K^t)^+ V \tag{44}$$

The LORETA calculates the forward problem by minimizing the cost function $\min J^t C J$ with the constraint of $V = KJ$ where $C = [BW]^t [BW]$ with B as discrete Laplacian operator to smooth the output and weight matrix W as:

$$W = \Omega \otimes I \tag{45}$$

Here I is identity matrix and $\otimes$ denotes the Kronecker product with Ω as $M \times M$ diagonal matrix with diagonal elements as:

$$\Omega_{ii} = \sqrt{\sum_{\alpha=1}^{N} k_{\alpha i}^T k_{\alpha i}} \tag{46}$$

Hence from the above derivations for different parameters related to LORETA solution, the current density can be predicted as:

$$J_{\text{LORETA}} = (C)^{-1} K^t [K C^{-1} K^t]^+ V \tag{47}$$

For this hybrid algorithm, the current density is calculated by using equation for WMN method. This vector J_{WMN} is used to build a weight matrix by the formula:

$$W_h = diag(J_{\text{WMN}}(i)) \tag{48}$$

Due to this new weight matrix, C_h is developed which is dependent upon the calculation made above with the help of WMN algorithm. Hence,

$$C_h = W_h B^t B W_h \tag{49}$$

In the end, the equation for the computation of current density for this new hybrid WMN–LORETA method is written as:

$$J_{\text{WMN-LORETA}} = (C_h)^{-1} K^t [K (C_h)^{-1} K^t]^+ V \tag{50}$$

This technique was examined by using 138 electrodes distributed on the scalp surface with 429 sources on the cerebral volume. The simulations for WMN, LORETA and Hybrid WMN–LORETA are shown for comparison between them. The comparison is done in terms of resolution matrix, i.e., $R = TK$. The resolution matrix for these methods is shown and it asserts the fact that this method has got near value to that of identity matrix which leads to ideal condition for less error and more accuracy. Also in terms of computational time, the suggested algorithm used less computing time than LORETA algorithm. Hence, the hybrid algorithm is efficient for solution of EEG inverse problem.

3.9. Recursive sLORETA–FOCUSS

Recursive sLORETA–FOCUSS combines the features of sLORETA and FOCUSS in a recursive manner to estimate the electrical activity inside the brain. The algorithm is presented and explained in [38] in such a way that it starts with the estimation of the current density by using sLORETA method. The current density is estimated by using sLORETA method given by:

$$J_{\text{sLORETA}} = \hat{S}_j \times J_{\text{MNE}} \tag{51}$$

where $\hat{S}_j$ the variance of the estimated current density and J_{MNE} is the current density estimation for minimum norm method. In the next step, weight matrix is constructed by using the mathematical relation given below:

$$W_i = PW_{i-1}[diag(\hat{J}_{i-1}(1), \hat{J}_{i-1}(2), \ldots, \hat{J}_{i-1}(3M))] \tag{52}$$

where $\hat{J}_{i-1}(n)$ is the nth element of vector $\hat{J}$ at the $(i-1)$th iteration. P is a diagonal matrix explained as:

$$P = diag[1/||K_1||, 1/||K_2||, \ldots, 1/||K_{3M}||] \tag{53}$$

This method is utilized for the calculation of the current density by using following equation:

$$\hat{J}_i = W_i W_i^t K^t (K W_i W_i^t K^t)^+ V \tag{54}$$

The FOCUSS is a recursive method for which the weight matrix is updated each time based on the data provided by the current density estimates of the previous ith iteration. This procedure is repeated (and hence the name recursive) to eliminate the non-active areas of brain. Hence after this elimination, new space is defined for active area only. The said steps are repeated until the so called convergence criterion is met. Here convergence means that the number of nodes in the newly defined solution space is less than the sensors used for measurements.

The algorithm is designed and analysed by simulation on MATLAB by assuming the presence of two current dipoles in the brain and a comparison is made between various localization algorithms such as sLORETA, FOCUSS, sLORETA–FOCUSS and recursive sLORETA–FOCUSS. According to the simulated images, sLORETA produced smooth and diffused reconstructed images for 2-dipoles which show the inability of sLORETA to localize the dipoles correctly. The FOCUSS algorithm alone provides sparse solution which does not suffice the need for a method to provide satisfactory localization results. The hybrid sLORETA–FOCUSS provides exact convergence to the real dipole with no localization error. However, the problem with this hybrid algorithm is of the generation of small fake sources beside the space solution. The recursive sLORETA–FOCUSS algorithm gives the best solution which provides exactly the same result as the simulated dipole. Upon making a comparison between the said four algorithms in terms of computational time taken for a method to localize the sources, the new designed hybrid algorithm, recursive sLORETA–FOCUSS is more time efficient as it takes 323.7031 s unlike the sLORETA–FOCUSS (330.4531 s) and FOCUSS (494.0313 s).

3.10. Shrinking LORETA–FOCUSS

This is hybrid algorithm combining the features of LORETA and FOCUSS algorithm in which the weight matrix is iterated along with the solution space which is also subjected to alteration during the localization. The idea is presented and explained in [39], as first computing the current density using LORETA algorithm as it provides smoothness and relatively small localization error. After this, weight matrix and the solution space is recursively iterated. The algorithm works by estimation of current density using LORETA algorithm. The weight matrix is calculated. After this a smoothing operator L is introduced such that:

$$L.\hat{J} = l_1^T, l_2^T, \ldots, l_M^T \tag{55}$$

Therefore, the smoothest current densities are:

$$l_i = \frac{1}{s_i + 1}\left(\hat{J}_i + \sum_u \hat{J}_u\right) \tag{56}$$

where s_i denotes the number of neighbouring nodes with the region defined by u.

The smoothing operator is used for the retaining of prominent nodes in estimated topography. The selection of prominent nodes is done by setting down threshold value in the topography. This process is repeated till the convergence. The results discussed in [12] show that the algorithm provides reconstruction of sources with relatively high spatial resolution as compared to LORETA algorithm. The localization capability is compared with other algorithms in terms of energy error (E_{enrg}) which is calculated as:

$$E_{\text{enrg}} = 1 - \frac{||\hat{J}_{\text{max}}||}{||J_{\text{simu}}||} \tag{57}$$

where $||\hat{J}_{\text{max}}||$ is the power of maxima in the estimated current density and $\left\|J_{\text{simu}}\right\|$ is the power of the simulated point source.

The results demonstrate that the mean localization error for this algorithm is low (0.72) as compared with LORETA (13.41) and LORETA–FOCUSS (2.33) algorithms. However, the energy error as defined above is also numerically smaller (0.73) if compared with LORETA (96.75) and LORETA–FOCUSS (8.44). However, this method is evaluated on simulated data only. The algorithm is not validated by using experimental data.

4. Summary

After having a detailed discussion of the methods provided so far in the literature, there arises a need to summarize the methods. The methods can be summarized by providing the author, year of invention, citations, advantages and limitations of the localization method. Table 1 shows that LORETA is the most popular method for the source localization as it has got maximum number of citations (1061) so far, though have got disadvantage of low spatial localization. However, the methods which are derived from LORETA such as sLORETA and eLORETA have got better localization capability but lagging in terms of resolution. The Hybrid weighted minimum norm method employ hybridization of minimum norm and LORETA with lot of system complexities in it which ultimately increases the required computational time. The shrinking LORETA–FOCUSS is also a hybrid algorithm taking advantage of LORETA and FOCUSS but is not validated experimentally. The other methods such as MUSIC and RAP-MUSIC method perform better in terms of localization but have problem of large computations, computational time and loss of data.

After the discussion of the various inverse methods, now we proceed to have a feature based comparison between various source localization methods.

Table 2 shows the comparison between different localization algorithms. This comparison is made by different Authors such as the one made between LORETA and Backus and Gilbert, minimum norm and WROP by Pascual in terms of localization error and estimated current density. It shows that the LORETA performs better than discussed algorithms. The other comparison is made between shrinking LORETA–FOCUSS and LORETA, LORETA–FOCUSS by He Sheng et al. in terms of energy and localization error and it is shown that the suggested algorithm has got comparatively low localization error resulting in better performance. The

Table 1
Summary of different techniques for solution of EEG inverse problem.

S. no.	Method	Author	Advantages	Limitations
1.	Minimum norm solution [25]	Matti S. Hamalainen (1984)	Provides good initial results in terms of resolution and current estimation.	Fails to address the issue of deep source localization and has got more localization error as compared to LORETA, WMN etc. with incapability of localizing non-boundary sources.
2.	LORETA [28]	R.D. Pascual (1994)	High localization capability for localizing of boundary sources and deep sources. Many variations are provided on this basic localization algorithm.	Has got low spatial resolution with blurred images. Low spatial resolution is undesirable in feature extraction of spatio-temporal pattern recognition. The regularization causes increase in spatial blurring.
3.	FOCUSS [29]	F. Irina et al. (1995)	Better Localization with capability to handle non-uniquely defined localized energy sources. Provides stable outputs.	Involves large mathematical calculations and hence large computational time due to continuous iteration of weight matrix.
4.	Recursive MUSIC, RAP MUSIC [30,31]	J.C. Mosher and R.M. Leahy (1998, 1999)	Recursive MUSIC along with its modifications provides better estimation with low localization error.	Model estimation includes random error and noise which causes difficulties for true signal estimation. Procedure includes weighted sum of errors and noises which increases complexity in the algorithm.
5.	sLORETA [33]	R.D. Pascual (2002)	Exact zero error localization as compared with minimum norm and Dale method.	Due to low resolution imaging method, it exhibits the poor performance in recovering multiple sources when the point-spread functions of sources overlap. Also, due to instability of EEG inverse problem, sometimes regularization is employed which causes increase in the spatial blurring of LORETA and sLORETA solutions.
6.	Shrinking LORETA–FOCUSS [39]	He Sheng (2003)	Provides better results in terms of minimized localization error as compared to LORETA, LORETA–FOCUSS algorithms.	Just evaluated on simulated data, not validated through real time data.
7.	Hybrid weighted minimum norm [32]	C.Y. Song et al. (2005)	Provides better estimation by using features of LORETA and WMN as iterations make the algorithm more accurate with in-depth localization and less erroneous.	The algorithm allows large computations and repeated iterations due to which computational time is high. Also the chance of loss of data is there due to continuous iterations of weight matrix.
8.	eLORETA [35]	R.D. Pascual (2007)	Standardized method with theoretical expected variance as Unity. Authentic localization technique with zero localization error.	The low resolution feature of eLORETA causes blurring in the images when the space is subjected to regularizations.
9.	WMN–LORETA [37]	R. Khemakhem et al. (2008)	Hybrid method with combined features of LORETA and WMN which provides better resolution than LORETA and WMN alone.	The system is complex and requires more computational time. System valid for localization highly active regions such as somatosensory evoked potentials.
10.	Recursive sLORETA–FOCUSS [38]	R. Khemakhem et al. (2008)	More efficient in terms of computational time and localization.	No validation provided. Results were produced on simulated data.

Table 2
Comparisons between various localization algorithms.

Author	Method	Compared with	Comments
R.D. Pascual-Marqui (1999)	LORETA	Backus and Gilbert, minimum norm, WROP	Results are compared in terms of localization error and estimated current density. LORETA performs well as compared to others.
He Sheng et al. (2003)	Shrinking LORETA–FOCUSS	LORETA, LORETA–FOCUSS	Comparison is made by using different parameters such as energy error, maximum localization error, maximum energy error, etc. The proposed algorithm shows better results on comparison.
R. Khemakhem et al. (2007)	sLORETA–FOCUSS	sLORETA, LORETA–FOCUSS	Localization error is compared for three methods that are sLORETA; LORETA–FOCUSS and sLORETA–FOCUSS. The localization error for the method suggested is less and improved for locating of simulated sources.
R. Khemakhem et al. (2008)	Recursive sLORETA–FOCUSS	sLORETA–FOCUSS, FOCUSS	Computing time and localization ability is compared for methods. The suggested algorithm is well suited for less computing time and more accurate results.
R. Khemakhem et al. (2008)	WMN–LORETA	WMN, LORETA	Computational time and resolution is compared. Suggested algorithm is more time efficient.

Table 3
Feature based comparison of methods.

S. no.	Method	Resolution	Complexity/computational time	Validation	Low localization error
1.	Minimum norm	×	√	√	×
2.	LORETA	×	√	√	×
3.	FOCUSS	×	×	√	NA
4.	Recursive MUSIC	NA	×	×	NA
5.	sLORETA	×	√	√	√
6.	Shrinking LORETA–FOCUSS	×	×	×	√
7.	Hybrid Weighted MN	×	×	×	NA
8.	eLORETA	×	√	√	√
9.	WMN–LORETA	×	×	×	√
10.	Recursive sLORETA–FOCUSS	×	√	×	√

sLORETA–FOCUSS technique is compared with individual sLORETA and hybrid LORETA–FOCUSS in terms of localization error. The other hybrid algorithms developed by Khemakhem et al. such as WMN–LORETA and recursive sLORETA–FOCUSS are compared with WMN, LORETA and sLORETA–FOCUSS and FOCUSS respectively with respect to Computational time, resolution and localization capability.

Before going into the section of challenges and discussion for the future studies and better implementation of EEG inverse problem, let us have a look at Table 3 which provides a quick overview of the methods used so far in terms of resolution, computational time (which corresponds to processing speed), validation and localization error. The Table 3 defines the relative features of different algorithms with the nomenclature as defined below.

The methods having low and high spatial resolution are shown with cross (×) and tick (√) signs, respectively. Same definition holds for the computational time as the methods having large iterations and repeated procedures are classified as non-favourable methods for this features so they are cross marked while remaining being efficient in terms of computational time are checked. There are some methods which are just inspected by using simulated data so they are classified as having no validation hence a cross is placed and vice versa. The last parameter is localization error which can be regarded as most important parameter for quality check for any localization method. The lower the localization error the better is the method. Here low localization error is compared for different algorithm relatively. NA in the table produced below means that the information so far is not available form literature to the authors.

5. Challenges and future recommendations

The EEG being an imaging technique provides good temporal resolution for the reflection of the neuronal activity but corresponds to poor spatial resolution which results in undesirable feature whenever it used for the source localization [40]. Hence for the solution of so called EEG Inverse problem, different methods are invented and explained by different researchers. However, as defined above the mathematical relations governing the methods, the results obtained, comparisons in terms of computational time taken, localization ability, localization error, energy error, system complexity, improved resolution are important parameters. It can be deduced from the above discussion that for the solution of ill posed EEG inverse problem, one should have following points in the mind:

(1) The designed algorithm should avoid the problem of not localizing the deep sources unlike the minimum norm solution.
(2) The algorithm should be developed with the ability of localizing the sources with better spatial resolution unlike the problems posed by LORETA (and its derived methods such as sLORETA, eLORETA). Because with low resolution, the recovery of multiple sources is difficult. Also due to ill-posedness of EEG inverse problem, one has to impart regularization methods. These regularization methods cause the increase in the blurring of images in LORETA family. So the method should be developed with improved spatial resolution.
(3) The issue of repeated iterations such as with some hybrid algorithms (LORETA–FOCUSS, sLORETA–FOCUSS, etc.) causes the system to become slow as the continuous iterations in the weight matrix takes the time and makes the system complex. This increased complexity sometimes results in the loss of information and induction of noise. So the computations should be minimized for reduction of time and calculations.
(4) The algorithm should be validated with real time data so as to confirm the results being produced unlike the results obtained by simulated data.
(5) The problem of dimension reduction should also be addressed.

Hence, from the above discussion, it is evident and clear that the methods applied so far have though provided good results and continuous improvement in the field of inverse problem solution is going on but still it requires coping with the issues of low resolution, system complexity, slow processing, results validations and stability of solution and localization error. Therefore, inverse problem needs to be solved with mentioned constraints so as to resolve issues related to applied neuroscience.

Acknowledgment

This work is funded by Graduate Assistance ship scheme provided by Universiti Teknologi PETRONAS, Perak, Malaysia. The authors are thankful to the Centre for Intelligent Signal and Imaging Research, Department of Electrical and Electronic Engineering, Universiti Teknologi PETRONAS, Tronoh, Perak, Malaysia for providing necessary facilities to conduct this research.

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