On the Covariant Gauge α of the Linearized Gravity in De Sitter Spacetime

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Abstract. In previous work [1, 2], we studied the linearized gravity with covariant gauge $\beta = 2/3$ and $\alpha = 5/3$. It was found that the sum of the source and initial contributions reproduces the correct field configuration over the whole de Sitter spacetime. In this paper, we extend this work to generalizing the linearized gravitational field in an arbitrary value of the gauge parameter α but the gauge parameter β remains the same.

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INTRODUCTION

De Sitter spacetime describes an inflationary phase of the very early universe after the Big-Bang singularity [3]. (See, e.g., [4–6] for early discussion on this subject). It is considered as a model of empty space universe with spatially flat section in which the dynamic of the universe is governed by the cosmological constant, which is thought to be dark energy in our universe. The recent observational data is strongly in favor to the universe at present may undergo a de Sitter phase of expansion [7, 8]. Mathematically, de Sitter spacetime can be constructed as a 4–dimensional one-sheeted hyperboloid $(\mathbf{R}^4, g_{ab})$ embedded in a 5–dimensional flat space $(\mathbf{R}^5, \eta_{\mu\nu})$ [3],

$$-(X^0)^2 + \sum_{i=1}^{4}(X^i)^2 = \frac{1}{H^2}\,, \tag{1}$$

where the Hubble constant H determines the expansion rate of the spatial section. The metric $\eta_{\mu\nu}$ and g_{ab} correspond to the 5–dimensional ambient space and 4–spacetime intrinsic coordinates in the de Sitter hyperboloid, respectively, where the Greek indices $\mu,\nu = 0,\cdots,4$ and the lowercase Latin indices $a,b = 0,\cdots,3$. This surface is related to the 10 parameters isometry group SO(4,1). Thus, de Sitter spacetime is a maximally symmetric 4–dimensional homogeneous and isotropic space that gives the solution of the vacuum Einstein field equation with positive cosmological constant $\Lambda = 3H^2$,

$$R_{ab} - \frac{1}{2}Rg_{ab} + 3H^2 g_{ab} = 0\,, \tag{2}$$

where R_{ab} and R are the corresponding Ricci tensor and Ricci scalar, respectively.

It has been a controversy that the field coming from a charge following a geodesic only fills the space of the union of the charge's future lightcone in the future event horizon, and thus violates the Gauss's Law because there are region such that the flux at sufficiently large sphere around the charge vanishes, e.g., in the region of the past event horizon of the charge [9–12]. This is due to the spacelike nature of past infinity of de Sitter spacetime that casts doubt to the validity of the usage of the covariant retarded Green's function to act as a tool to generate the field causally from a source. This paradox can be resolved by including not only the source term but also the initial data on the Cauchy hypersurface at infinite past when calculating the field configuration using retarded Green's function [1, 2, 13]. Therefore the field is correctly reproduced to satisfy Gauss's Law at any instance of time over the Cauchy hypersurface.

In previous work [1] we show the correct field configuration is reproduced from the sums of source term along its geodesic and initial data at past infinity using retarded Green's function with the gauge condition

$$\nabla_a h^{ab} - \frac{5}{2}\nabla^b h = 0\,, \tag{3}$$

and the gauge parameter $\alpha = 5/3$. In this paper we extend the work for linearized gravity by generalizing the field with the gauge parameter α arbitrary. The remainder of the paper is organized as follows. We briefly introduce and recall some essential facts concerning the structure of de Sitter space as a preliminary in the next section. We then formulate the retarded Green's function by generalizing the field in a one-parameter family of covariant gauge α in addition to the one with gauge fixed at

$\alpha = 5/3$, which will be the subject of this paper, in the following of next section. In the final section we summarize this paper. We use natural units $\hbar = c = 16\pi G = 1$ and the metric signature $(-+++)$ throughout this paper.

DE SITTER SPACE

In de Sitter spacetime, the choice of the vacuum state is not unique in general due to lacking of global timelike Killing vector field [14], K^a, satisfying

$$\nabla_{(a}K_{b)} = 0 \,. \tag{4}$$

However, we can find a coordinate system due to high symmetry of this spacetime in which the spacetime is static, i.e., the metric is time-independence. In such a coordinate system a natural vacuum state exists for the scalar field theory, called the Euclidean or Bunch-Davies vacuum state [15]. This static form can be constructed if the parametrization is taken as

$$\begin{aligned} X^0 &= \sqrt{H^{-2}-R^2}\sinh HT \,, && (5)\\ X^1 &= \sqrt{H^{-2}-R^2}\cosh HT \,, && (6)\\ X^2 &= R\cos\theta \,, && (7)\\ X^3 &= R\sin\theta\cos\phi \,, && (8)\\ X^4 &= R\sin\theta\sin\phi \,, && (9) \end{aligned}$$

where $T \in (-\infty,\infty)$, $R \in [0,H^{-1})$, $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$. The corresponding form of the metric is

$$ds^2 = -(1-H^2R^2)dT^2 + (1-H^2R^2)^{-1}dR^2 + R^2 d\Omega_2 \,, \tag{10}$$

where $d\Omega_2$ is the metric of unit 2-sphere, i.e.,

$$d\Omega_2 = d\theta^2 + \sin^2\theta d\phi^2 \,. \tag{11}$$

This coordinate system covers only the quarter of the whole manifold of de Sitter space as illustrated in the region labeled as I in fig. (1). At any instant of time the origin[1] (North Pole) of the spatial section lies along the vertical right edge at $R = 0$ and the coordinate time T increases from $-\infty$ to $+\infty$ towards the future. Due to the metric is time independence, this coordinate system admits a future directed timelike Killing vector field $(\partial/\partial T)^a$. Notice that the vertical hyperboloidal curve ($R =$ constant) with arrow pointing upward is the integral curves of the timelike Killing vector $K^a = (\partial/\partial T)^a$. In this expanding quarter of de Sitter space, the radius R can be extended to the null hypersurface, $R = H^{-1}$, which

[1] Without loss of generality, the origin can be taken to be any point of de Sitter space.

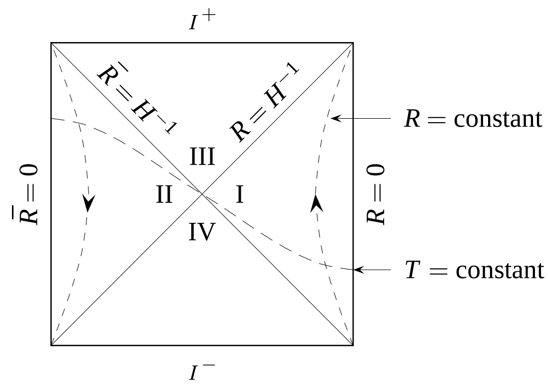

FIGURE 1. The Carter-Penrose diagram for static coordinate system of de Sitter spacetime. Each point in this diagram represents a 2-sphere spanned by θ,ϕ. The $\mathcal{I}^-$ and $\mathcal{I}^+$ represent the past and future infinity. These conformal infinities are spacelike in nature.

we called the cosmological horizon of the de Sitter universe. It is possible to extend the coverage beyond the limit $R = H^{-1}$ by the coordinate chart of Eqs. (5) and (6) with the factor $\sqrt{H^{-2}-R^2}$ replaced by $-\sqrt{H^{-2}-\bar{R}^2}$. This chart covers the region II in fig. (1) where $\bar{R}$ is the radius originated from the antipodal[2] origin (South Pole) of the spatial section lying along the vertical left edge ($\bar{R} = 0$) and the coordinate time T increases from $-\infty$ to $+\infty$ towards the past. In this region, the vertical hyperboloidal curve ($\bar{R} =$ constant) with arrow pointing downward is the integral curves of the timelike Killing vector field, i.e., this region contains the past directed timelike Killing vector field.

The spacelike hypersurface, Σ, is a 3-sphere of constant time ($T =$ constant) evolves along the way whose normal vector coincides with the future directed Killing vector in region I and past directed Killing vector in region II as depicted in fig. (1) when the common coordinate time T runs from negative infinity to positive infinity. If a point mass M is placed at the North Pole, one must find a point mass M exists in the South Pole on the hypersurface Σ. This is related to the fact that the total conserved charge corresponding to a de Sitter boost symmetry must vanish [16, 17], i.e., the stress-energy tensor T^{ab} must satisfy

$$\int_\Sigma d\Sigma^a T_{ab} K^b = 0 \,, \tag{12}$$

[2] Given a point x located by the $X^\mu(x)$, an antipodal point is a point $\bar{x}$ located by the $-X^\mu(\bar{x})$, so $X^\mu(\bar{x}) = -X^\mu(x)$.

where the integration is performed over the spatial hypersurface Σ in which $d\Sigma^a = d\Sigma \times t^a$, where t^a is the future directed timelike unit vector normal to the surface Σ.

GAUGE GENERALIZATION IN THE GRAVITATIONAL FIELD

The Lagrangian density describing the massless spin–2 field of pure gravity with a positive cosmological constant is given by

$$\mathcal{L}_{\text{full}} = \sqrt{-\tilde{g}}(\tilde{R} - 2\Lambda)\,, \tag{13}$$

where $\tilde{g} = \det \tilde{g}_{ab}$ is the determinant of the full metric and $\tilde{R}$ is the corresponding Ricci scalar. When a small perturbation h_{ab} in the background metric g_{ab} is introduced, the full metric can be written as $\tilde{g}_{ab} = g_{ab} + h_{ab}$. We have for linearized gravity, by expanding the Lagrangian density to second order in h_{ab}, can be expressed in the form

$$\begin{aligned}\mathcal{L}^{(2)} + \mathcal{L}_{\text{gf}} \quad = \quad & \tfrac{1}{2}\sqrt{-g}\Big\{(\mathcal{T} + \mathcal{T}_{\text{gf}})^{abcdef}\nabla_a h_{bc} \nabla_d h_{ef} \\ & + \mathcal{S}^{abcd} H^2 h_{ab} h_{cd}\Big\}\,,\end{aligned} \tag{14}$$

where $(\mathcal{T} + \mathcal{T}_{\text{gf}})^{abcdef}$ and $\mathcal{S}^{abcd}$ are tensors formed by the metrics, and $\mathcal{L}_{\text{gf}} \propto \mathcal{T}_{\text{gf}}{}^{abcdef}$ is the Lagrangian density contributed by the gauge fixing term (as referred to Eq. (3)). Then, the conjugate momentum current is given by

$$\pi^{abc} = (\mathcal{T} + \mathcal{T}_{\text{gf}})^{a(bc)def} \nabla_d h_{ef}\,. \tag{15}$$

We note that the conjugate momentum current is a gauge invariant quantity. The covariant graviton propagator $Q_{aba'b'}(x, x')$ satisfying the equation of motion operator

$$\begin{aligned}\Big[(\mathcal{T} + \mathcal{T}_{\text{gf}})^{cabdef} \nabla_c \nabla_d - H^2 S^{abef}\Big] Q_{efa'b'}(x, x') \\ = \delta^{ab}{}_{a'b'}(x, x')\,,\end{aligned} \tag{16}$$

was obtained in [18] for the gauge $\beta = 2/3$, where the scalar sector does not increase with distance. The retarded Green's function is found as the difference of the values of the propagators between two points to its antipodal points counterpart. In the rest of this paper we drop the argument in the equation for simplicity. It is easy to restore the argument from a given tensor as the unprimed index refers to the point at x and the primed index refers to the point at x'.

Let us consider the gravitational field h_{ab} coupled to the stress-energy tensor T^{ab}. The field in the future domain of dependence of the Cauchy surface Σ, denoted by $D^+(\Sigma)$, is given in terms of the retarded Green's function as [1, 2]

$$h_{ab} = h^{(S)}_{ab} + h^{(I)}_{ab}\,, \tag{17}$$

where $h^{(S)}_{ab}$ is called the source field (the field generated by the source)

$$h^{(S)}_{ab} = \int_{D^+(\Sigma)} d^4x' \sqrt{-g'}\, G_{aba'b'} T^{a'b'}\,, \tag{18}$$

and $h^{(I)}_{ab}$ is called the initial field (the field generated by the initial surface)

$$h^{(I)}_{ab} = \int_\Sigma d\Sigma_{c'} \Big[\pi^{c'a'b'} G_{aba'b'} - (L_\pi G)_{ab}{}^{c'a'b'} h_{a'b'}\Big]\,. \tag{19}$$

Here, g' is the determinant of the metric at x' and

$$(L_\pi G)_{ab}{}^{c'a'b'} = (\mathcal{T} + \mathcal{T}_{\text{gf}})^{c'(a'b')d'e'f'} \nabla_{d'} G_{abe'f'}\,. \tag{20}$$

The field $h_{a'b'}$ and the corresponding conjugate momentum current $\pi^{c'a'b'}$ adopted at infinite past surface Σ is the input field identical to the field reproduced in de Sitter space. This is because the initial surface is required to satisfy the Gauss Law in the first place.

In [1] we adopted the gauge $\alpha = 5/3$ in calculating the field given by Eq. (17). In the rest of this section we restore the gauge α in the field in addition to the field at $\alpha = 5/3$. As was found in [18] that the coefficients of each bi-tensor term is proportional to $\sum_i (A_i + B_i\alpha) f_i(z)$, where A_i and B_i are arbitrary constants. Now, the constant coefficients can be written as

$$A_i + B_i\alpha = \left(A_i + \frac{5}{3}B_i\right) - \frac{5}{3}B_i\left(1 - \frac{3\alpha}{5}\right). \tag{21}$$

We see that the first term on the RHS is the coefficient at which the gauge $\alpha = 5/3$ is used. In a similar manner, the retarded Green's function can be written as

$$G_{aba'b'} = G_{aba'b'}\Big|_{\alpha=5/3} + \left(1 - \frac{3\alpha}{5}\right)\widetilde{G}_{aba'b'}\,, \tag{22}$$

where

$$\widetilde{G}_{aba'b'} = G_{aba'b'}\Big|_{\alpha=-5/3,\,\forall A_i=0}\,. \tag{23}$$

The coefficients of the operator L_π take the form $A_i + B_i\alpha^{-1}$, and similarly the operator can be written as

$$L_\pi = L_\pi\Big|_{\alpha=5/3} + \left(1 - \frac{5}{3\alpha}\right)\widetilde{L}_\pi\,, \tag{24}$$

where

$$\widetilde{L}_\pi = L_\pi\Big|_{\alpha=-5/3,\,\forall A_i=0}\,. \tag{25}$$

Thus, the second term of initial field in Eq. (19) reads

$$\begin{aligned} & (L_\pi G)_{ab}{}^{a'b'c'} \\ = \quad & (L_\pi G)_{ab}{}^{a'b'c'}\Big|_{\alpha=5/3} + \left(1 - \frac{3\alpha}{5}\right)(L_\pi \widetilde{G})_{ab}{}^{a'b'c'} \\ & + \left(1 - \frac{5}{3\alpha}\right)\left(\widetilde{L}_\pi G\Big|_{\alpha=5/3}\right)_{ab}{}^{a'b'c'}\,.\end{aligned} \tag{26}$$

Using the fact that $G|_{\alpha=5/3} + \widetilde{G} = G|_{\alpha=0}$ and

$$\left(\nabla_{a'} G_{ab}{}^{a'b'} - \frac{5}{2}\nabla^{b'} G_{ab}\right)\Big|_{\alpha=0} = 0\,, \qquad (27)$$

we find that

$$\begin{aligned}(L_\pi G)_{ab}{}^{c'a'b'} &= (L_\pi G)_{ab}{}^{c'a'b'}\Big|_{\alpha=5/3} \\ &\quad + \left(1 - \frac{3\alpha}{5}\right)\left(L_\pi|_{\alpha\to\infty}\widetilde{G}\right)_{ab}{}^{c'a'b'}\,.\end{aligned} \qquad (28)$$

Thus, we have for the gravitational field

$$h_{ab} = h_{ab}\Big|_{\alpha=5/3} + \left(1 - \frac{3\alpha}{5}\right)\left(\widetilde{h}^{(S)}_{ab} + \widetilde{h}^{(I)}_{ab}\right), \qquad (29)$$

where

$$\widetilde{h}^{(S)}_{ab} = \int_{D^+(\Sigma)} d^4x' \sqrt{-g'}\,\widetilde{G}_{aba'b'} T^{a'b'}\,, \qquad (30)$$

$$\begin{aligned}\widetilde{h}^{(I)}_{ab} &= \int_\Sigma d\Sigma_{c'}\Big[\pi^{c'a'b'}\widetilde{G}_{aba'b'} \\ &\quad - \left(L_\pi|_{\alpha\to\infty}\widetilde{G}\right)_{ab}{}^{c'a'b'} h_{a'b'}\Big]\,.\end{aligned} \qquad (31)$$

We note that the equations above are α invariant. In view of the constant coefficient defining the Eqs. (23), (29) and (30) imply that the correct field h_{ab} to be produced by the source is given by

$$h^{(S)}_{ab} = h^{(S)}_{ab}\Big|_{\alpha=0} - \frac{3\alpha}{5}\widetilde{h}^{(S)}_{ab}\,. \qquad (32)$$

This show that the extra term (the second term on the RHS of the equation above), which does not satisfy the gauge condition (3), possessed in the source field is a pure gauge field when the α is restored. The first term in this equation is identical to the input field at the initial surface. The initial field can be generalized to

$$h^{(I)}_{ab} = h^{(I)}_{ab}\theta(\Delta\chi - \Delta\tau) + \frac{3\alpha}{5}h^{(I)}_{ab}\Big|_{\alpha=5/3}\theta(\Delta\tau - \Delta\chi)\,, \quad (33)$$

where $\theta(X-Y)$ is the Heaviside step function, i.e., $\theta(X-Y) = 1$ if $X \geq Y$, otherwise vanishes. The first term on the RHS of the initial field, which is α invariant, is produced in the causal past of the point mass and the second term, which has an overall gauge factor, is produced in the causal future of the same mass. This second term canceled exactly with the extra term from the source field in the causal future of the point mass.

CONCLUSIONS

In this paper, we generalize the gravitational field produced by the source term and the initial surface in the covariant gauge α in de Sitter spacetime. In addition to the field at the fixed gauge, a term arose proportional to the α at which this term will vanish when the gauge $\alpha = 5/3$ is applied. The extra terms arising in the field that differ from the correct field is a pure gauge field. This field cancels the initial contribution in the causal future of the point mass along the geodesic. Thus, the sum $h^{(S)}_{ab} + h^{(I)}_{ab}$ is α independent and Eq. (17) reproduces the correct field configuration over the whole de Sitter spacetime.

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